A Sextuply Self-Orthogonal Latin Square of Order Nine and their Magic Squares

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Abstract

E. Nemeth [1)] coined the term *"self-orthogonal Latin square"* to denote a Latin square orthogonal to its *transpose*. Kichul Kim and V. K. Prasanna Kumar [2)] coined the term *"doubly self-orthogonal Latin square"* to denote a Latin square orthogonal to its *transpose* and to its *antitranspose;* by making to note that a doubly self-orthogonal Latin square is also a diagonal Latin square. In [3)] was proposed the term *"quadruply self-orthogonal Latin square"* to denote a Latin square orthogonal to its *transpose*, to its *antitranspose*, to its *rotation* around the axis "*x*" and to its *rotation* around the axis "*y*", by making to note that the *pandiagonal* Latin squares of order n = 6k ± 1 they are quadruply self-orthogonal Latin squares. In this paper two examples of *"sextuply self-orthogonal Latin squares"* of order nine are shown.-

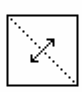
Quadruply Self-Orthogonal Latin Square

A Latin square is quadruply self-orthogonal if it is orthogonal to its transpose, to its antitranspose, to its rotation around the axis *"x"* and to its rotation around the axis *"y"*.

An example of order five:

a	b	c	d	e
c	d	e	a	b
e	a	b	c	d
b	c	d	e	a
d	e	a	b	c

LS_1



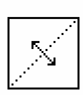
a	c	e	b	d
b	d	a	c	e
c	e	b	d	a
d	a	c	e	b
e	b	d	a	c

transpose LS_1

aa	bc	ce	db	ed
cb	dd	ea	ac	be
ec	ae	bb	cd	da
bd	ca	dc	ee	ab
de	eb	ad	ba	cc

a	b	c	d	e
c	d	e	a	b
e	a	b	c	d
b	c	d	e	a
d	e	a	b	c

LS_1



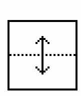
c	a	d	b	e
b	e	c	a	d
a	d	b	e	c
e	c	a	d	b
d	b	e	c	a

antitranspose LS_1

ac	ba	cd	db	ee
cb	de	ec	aa	bd
ea	ad	bb	ce	dc
be	cc	da	ed	ab
dd	eb	ae	bc	ca

a	b	c	d	e
c	d	e	a	b
e	a	b	c	d
b	c	d	e	a
d	e	a	b	c

LS_1



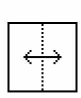
d	e	a	b	c
b	c	d	e	a
e	a	b	c	d
c	d	e	a	b
a	b	c	d	e

x rotation LS_1

ad	be	ca	db	ec
cb	dc	ed	ae	ba
ee	aa	bb	cc	dd
bc	cd	de	ea	ab
da	eb	ac	bd	ce

a	b	c	d	e
c	d	e	a	b
e	a	b	c	d
b	c	d	e	a
d	e	a	b	c

LS_1



e	d	c	b	a
b	a	e	d	c
d	c	b	a	e
a	e	d	c	b
c	b	a	e	d

y rotation LS_1

ae	bd	cc	db	ea
cb	da	ee	ad	bc
ed	ac	bb	ca	de
ba	ce	dd	ec	ab
dc	eb	aa	be	cd

Pandiagonal Squares of Order Three

In [4], the nine pandiagonal squares of order three were presented as follows:

7	8	9		5	6	4		3	1	2
1	2	3		8	9	7		6	4	5
4	5	6		2	3	1		9	7	8
6	4	5		1	2	3		8	9	7
9	7	8		4	5	6		2	3	1
3	1	2		7	8	9		5	6	4
2	3	1		9	7	8		4	5	6
5	6	4		3	1	2		7	8	9
8	9	7		6	4	5		1	2	3

In his website [5], Harvey Heinz wrote on this presentation: *"The 3 x 3 squares are the nine ways these numbers may be arranged in an array with all diagonals summing to 15. However, most of the rows and columns do not, so the squares are not magic. The nine arrays are arranged to form an order-9 simple magic square. The center number of each 3 x 3 array form one order-3 simple magic square ! "*

Diagonal Latin Square

A *Latin square* is *diagonal* if no symbol appears more than once in each one of both main diagonals. The above array is a *diagonal Latin square*:

7	8	9	5	6	4	3	1	2
1	2	3	8	9	7	6	4	5
4	5	6	2	3	1	9	7	8
6	4	5	1	2	3	8	9	7
9	7	8	4	5	6	2	3	1
3	1	2	7	8	9	5	6	4
2	3	1	9	7	8	4	5	6
5	6	4	3	1	2	7	8	9
8	9	7	6	4	5	1	2	3

A

According to the definition of Kichul Kim and Prasanna Kumar, **A** it is a *perfect Latin square* because no symbol appears more than once in any main subsquare; and:

It is sextuply self-orthogonal !!!

Sextuply Self-Orthogonal Latin Square

A Latin square is "*sextuply self-orthogonal"* if it is orthogonal to its transpose, to its antitranspose, to its rotation around the axis *"x"*, to the transpose of its rotation around the axis *"x"*, to its rotation around the axis *"y"* and to the transpose of its rotation around the axis *"y"*.

Magic Squares

In this paper, the sextuple orthogonality of **A** is shown by means of the six magic squares with the consecutive numbers of 0 to 80 by using the Eulerian formula and **B** = **A**(i, j , k-1):

$$9\begin{vmatrix}6&7&8&4&5&3&2&0&1\\0&1&2&7&8&6&5&3&4\\3&4&5&1&2&0&8&6&7\\5&3&4&0&1&2&7&8&6\\8&6&7&3&4&5&1&2&0\\2&0&1&6&7&8&4&5&3\\1&2&0&8&6&7&3&4&5\\4&5&3&2&0&1&6&7&8\\7&8&6&5&3&4&0&1&2\end{vmatrix}+\begin{vmatrix}6&0&3&5&8&2&1&4&7\\7&1&4&3&6&0&2&5&8\\8&2&5&4&7&1&0&3&6\\4&7&1&0&3&6&8&2&5\\5&8&2&1&4&7&6&0&3\\3&6&0&2&5&8&7&1&4\\2&5&8&7&1&4&3&6&0\\0&3&6&8&2&5&4&7&1\\1&4&7&6&0&3&5&8&2\end{vmatrix}=\begin{vmatrix}60&63&75&41&53&29&19&4&16\\7&10&22&66&78&54&47&32&44\\35&38&50&13&25&1&72&57&69\\49&34&37&0&12&24&71&74&59\\77&62&65&28&40&52&15&18&3\\21&6&9&56&68&80&43&46&31\\11&23&8&79&55&67&30&42&45\\36&48&33&26&2&14&58&70&73\\64&76&61&51&27&39&5&17&20\end{vmatrix}$$

B $\qquad$ $\mathbf{B}^t$ $\qquad$ magic square **1**

$$9\begin{vmatrix}6&7&8&4&5&3&2&0&1\\0&1&2&7&8&6&5&3&4\\3&4&5&1&2&0&8&6&7\\5&3&4&0&1&2&7&8&6\\8&6&7&3&4&5&1&2&0\\2&0&1&6&7&8&4&5&3\\1&2&0&8&6&7&3&4&5\\4&5&3&2&0&1&6&7&8\\7&8&6&5&3&4&0&1&2\end{vmatrix}+\begin{vmatrix}2&8&5&3&0&6&7&4&1\\1&7&4&5&2&8&6&3&0\\0&6&3&4&1&7&8&5&2\\4&1&7&8&5&2&0&6&3\\3&0&6&7&4&1&2&8&5\\5&2&8&6&3&0&1&7&4\\6&3&0&1&7&4&5&2&8\\8&5&2&0&6&3&4&1&7\\7&4&1&2&8&5&3&0&6\end{vmatrix}=\begin{vmatrix}56&71&77&39&45&33&25&4&10\\1&16&22&68&74&62&51&30&36\\27&42&48&13&19&7&80&59&65\\49&28&43&8&14&20&63&78&57\\75&54&69&34&40&46&11&26&5\\23&2&17&60&66&72&37&52&31\\15&21&0&73&61&67&32&38&53\\44&50&29&18&6&12&58&64&79\\70&76&55&47&35&41&3&9&24\end{vmatrix}$$

$\mathbf{B}$ $\qquad$ $\mathbf{B}^{at}$ $\qquad$ magic square **2**

$$9\begin{vmatrix}6&7&8&4&5&3&2&0&1\\0&1&2&7&8&6&5&3&4\\3&4&5&1&2&0&8&6&7\\5&3&4&0&1&2&7&8&6\\8&6&7&3&4&5&1&2&0\\2&0&1&6&7&8&4&5&3\\1&2&0&8&6&7&3&4&5\\4&5&3&2&0&1&6&7&8\\7&8&6&5&3&4&0&1&2\end{vmatrix}+\begin{vmatrix}7&8&6&5&3&4&0&1&2\\4&5&3&2&0&1&6&7&8\\1&2&0&8&6&7&3&4&5\\2&0&1&6&7&8&4&5&3\\8&6&7&3&4&5&1&2&0\\5&3&4&0&1&2&7&8&6\\3&4&5&1&2&0&8&6&7\\0&1&2&7&8&6&5&3&4\\6&7&8&4&5&3&2&0&1\end{vmatrix}=\begin{vmatrix}61&71&78&41&48&31&18&1&11\\4&14&21&65&72&55&51&34&44\\28&38&45&17&24&7&75&58&68\\47&27&37&6&16&26&67&77&57\\80&60&70&30&40&50&10&20&0\\23&3&13&54&64&74&43&53&33\\12&22&5&73&56&63&35&42&52\\36&46&29&25&8&15&59&66&76\\69&79&62&49&32&39&2&9&19\end{vmatrix}$$

$\mathbf{B}$ $\qquad$ $\mathbf{B}^{x}$ $\qquad$ magic square **3**

$$9\begin{vmatrix}6&7&8&4&5&3&2&0&1\\0&1&2&7&8&6&5&3&4\\3&4&5&1&2&0&8&6&7\\5&3&4&0&1&2&7&8&6\\8&6&7&3&4&5&1&2&0\\2&0&1&6&7&8&4&5&3\\1&2&0&8&6&7&3&4&5\\4&5&3&2&0&1&6&7&8\\7&8&6&5&3&4&0&1&2\end{vmatrix}+\begin{vmatrix}7&4&1&2&8&5&3&0&6\\8&5&2&0&6&3&4&1&7\\6&3&0&1&7&4&5&2&8\\5&2&8&6&3&0&1&7&4\\3&0&6&7&4&1&2&8&5\\4&1&7&8&5&2&0&6&3\\0&6&3&4&1&7&8&5&2\\1&7&4&5&2&8&6&3&0\\2&8&5&3&0&6&7&4&1\end{vmatrix}=\begin{vmatrix}61&67&73&38&53&32&21&0&15\\8&14&20&63&78&57&49&28&43\\33&39&45&10&25&4&77&56&71\\50&29&44&6&12&18&64&79&58\\75&54&69&34&40&46&11&26&5\\22&1&16&62&68&74&36&51&30\\9&24&3&76&55&70&35&41&47\\37&52&31&23&2&17&60&66&72\\65&80&59&48&27&42&7&13&19\end{vmatrix}$$

$\mathbf{B}$ $\qquad$ $\mathbf{B}^{xt}$ $\qquad$ magic square **4**

$$9\begin{vmatrix}6&7&8&4&5&3&2&0&1\\0&1&2&7&8&6&5&3&4\\3&4&5&1&2&0&8&6&7\\5&3&4&0&1&2&7&8&6\\8&6&7&3&4&5&1&2&0\\2&0&1&6&7&8&4&5&3\\1&2&0&8&6&7&3&4&5\\4&5&3&2&0&1&6&7&8\\7&8&6&5&3&4&0&1&2\end{vmatrix}+\begin{vmatrix}1&0&2&3&5&4&8&7&6\\4&3&5&6&8&7&2&1&0\\7&6&8&0&2&1&5&4&3\\6&8&7&2&1&0&4&3&5\\0&2&1&5&4&3&7&6&8\\3&5&4&8&7&6&1&0&2\\5&4&3&7&6&8&0&2&1\\8&7&6&1&0&2&3&5&4\\2&1&0&4&3&5&6&8&7\end{vmatrix}=\begin{vmatrix}55&63&74&39&50&31&26&7&15\\4&12&23&69&80&61&47&28&36\\34&42&53&9&20&1&77&58&66\\51&35&43&2&10&18&67&75&59\\72&56&64&32&40&48&16&24&8\\21&5&13&62&70&78&37&45&29\\14&22&3&79&60&71&27&38&46\\44&52&33&19&0&11&57&68&76\\65&73&54&49&30&41&6&17&25\end{vmatrix}$$

$\mathbf{B}$ $\qquad$ $\mathbf{B}^{y}$ $\qquad$ magic square **5**

$$9\begin{vmatrix}6&7&8&4&5&3&2&0&1\\0&1&2&7&8&6&5&3&4\\3&4&5&1&2&0&8&6&7\\5&3&4&0&1&2&7&8&6\\8&6&7&3&4&5&1&2&0\\2&0&1&6&7&8&4&5&3\\1&2&0&8&6&7&3&4&5\\4&5&3&2&0&1&6&7&8\\7&8&6&5&3&4&0&1&2\end{vmatrix}+\begin{vmatrix}1&4&7&6&0&3&5&8&2\\0&3&6&8&2&5&4&7&1\\2&5&8&7&1&4&3&6&0\\3&6&0&2&5&8&7&1&4\\5&8&2&1&4&7&6&0&3\\4&7&1&0&3&6&8&2&5\\8&2&5&4&7&1&0&3&6\\7&1&4&3&6&0&2&5&8\\6&0&3&5&8&2&1&4&7\end{vmatrix}=\begin{vmatrix}55&67&79&42&45&30&23&8&11\\0&12&24&71&74&59&49&34&37\\29&41&53&16&19&4&75&60&63\\48&33&36&2&14&26&70&73&58\\77&62&65&28&40&52&15&18&3\\22&7&10&54&66&78&44&47&32\\17&20&5&76&61&64&27&39&51\\43&46&31&21&6&9&56&68&80\\69&72&57&50&35&38&1&13&25\end{vmatrix}$$

$\mathbf{B}$ $\qquad$ $\mathbf{B}^{yt}$ $\qquad$ magic square **6**

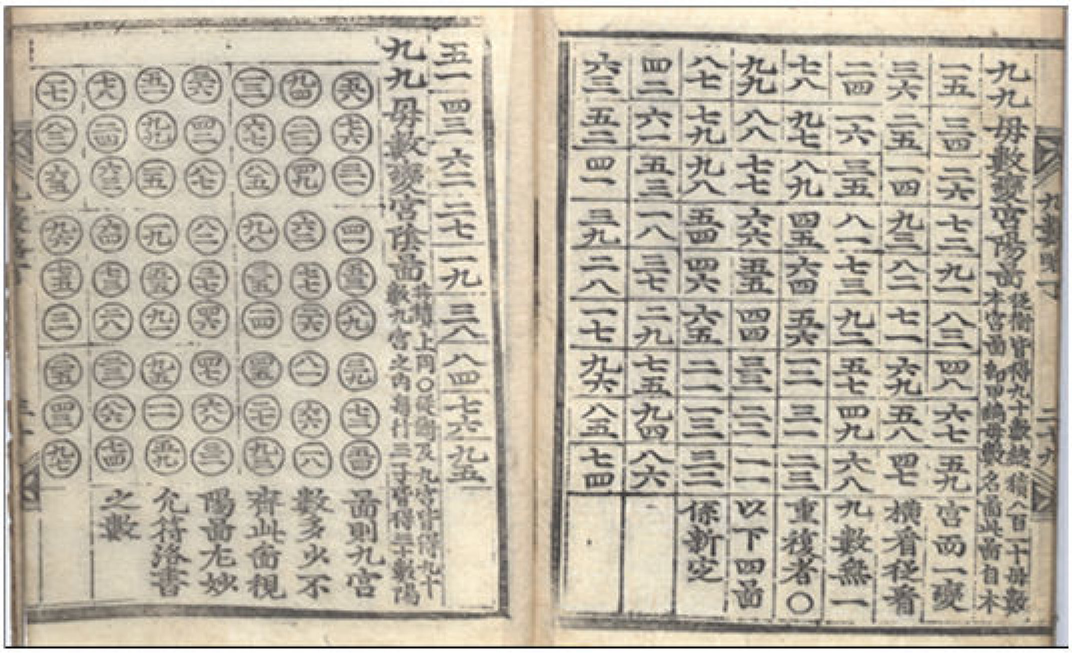
Appendix

$$E=\left[\begin{array}{ccc|ccc|ccc}0&3&6&1&4&7&2&5&8\\2&5&8&0&3&6&1&4&7\\1&4&7&2&5&8&0&3&6\\\hline 3&6&0&4&7&1&5&8&2\\5&8&2&3&6&0&4&7&1\\4&7&1&5&8&2&3&6&0\\\hline 6&0&3&7&1&4&8&2&5\\8&2&5&6&0&3&7&1&4\\7&1&4&8&2&5&6&0&3\end{array}\right]$$

Kichul Kim and Prasanna Kumar defined the Latin square **E** as *doubly self-orthogonal* and *perfect*, but they didn't identify the *sextuple orthogonality* of such a square.-.

Orthogonal Latin Squares of Choi Seok-Jeong

Recently in [6] a magic square of order nine constructed by using an orthogonal pair of Latin squares at least 67 years earlier than Leonard Euler by Choi Seok-Jeong (1646-1715) was presented**:**



5,1	6,3	4,2	8,7	9,9	7,8	2,4	3,6	1,5
4,3	5,2	6,1	7,9	8,8	9,7	1,6	2,5	3,4
6,2	4,1	5,3	9,8	7,7	8,9	3,5	1,4	2,6
2,7	3,9	1,8	5,4	6,6	4,5	8,1	9,3	7,2
1,9	2,8	3,7	4,6	5,5	6,4	7,3	8,2	9,1
3,8	1,7	2,9	6,5	4,4	5,6	9,2	7,1	8,3
8,4	9,6	7,5	2,1	3,3	1,2	5,7	6,9	4,8
7,6	8,5	9,4	1,3	2,2	3,1	4,9	5,8	6,7
9,5	7,4	8,6	3,2	1,1	2,3	6,8	4,7	5,9

37	48	27	70	81	62	13	24	5
30	38	46	63	71	79	6	14	22
47	28	39	80	61	72	23	4	15
16	27	8	40	51	32	64	75	56
9	17	25	33	41	49	57	65	73
26	7	18	50	31	42	74	55	66
67	78	59	10	21	2	43	54	35
60	68	76	3	11	19	36	44	52
77	58	69	20	1	12	53	34	45

Observe that each Latin square of the orthogonal pair is the rotation around the axis "y" of the other one.-

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