No Magic Franklin Square of Order n = 8k + 4

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Definition of Franklin Square

1) The entries are: $1, 2, 3, \ldots n^2$.-

2) The entries of every row and column add to a common sum called the magic sum:

$$Ms_{(n)} = (n^3 + n)/2$$

3) The entries in every half-row and half-column add to half the magic sum:

$$Hrc_{(n)} = (n^3 + n)/4$$

4) The entries of the main bent diagonals and all the bent diagonals parallel to them, add the magic sum:

$$Bd_{(n)} = Ms_{(n)}$$

5) The adjacent entries of every 2×2 sub-squares add the sum:

$$2x2Ssq_{(n)} = 2(n^2 + 1)$$

8x8 Franklin Squares

Daniel Schindel, Matthew Rempel and Peter Loly [1] determined that the *8x8 Franklin Squares* has 1.105.920 solutions; with 737.280 *Semi-Magic* (every main diagonal don't add the magic sum) and 368.640 *Magic* (every main diagonal add the magic sum). The 368.640 *Magic* are also *Pandiagonal* (every secondary diagonal add the magic sum); solutions that I corroborated in 2006 [6].-

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Semi-Magic [2]

1	46	51	32	35	62	17	16
60	23	10	37	26	7	44	53
14	33	64	19	48	49	30	3
55	28	5	42	21	12	39	58
9	38	59	24	43	54	25	8
63	20	13	34	29	4	47	50
6	41	56	27	40	57	22	11
52	31	2	45	18	15	36	61

Magic [6]

16x16 Franklin Squares

For the *16x16 Franklin Squares*, also has been obtained *Semi-Magic* and *Magic* solutions:

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

Semi-Magic [2]

1	160	226	127	227	126	4	157	161	224	66	63	67	62	164	221
252	101	27	134	26	135	249	104	92	37	187	198	186	199	89	40
29	132	254	99	255	98	32	129	189	196	94	35	95	34	192	193
232	121	7	154	6	155	229	124	72	57	167	218	166	219	69	60
9	152	234	119	235	118	12	149	169	216	74	55	75	54	172	213
244	109	19	142	18	143	241	112	84	45	179	206	178	207	81	48
21	140	246	107	247	106	24	137	181	204	86	43	87	42	184	201
240	113	15	146	14	147	237	116	80	49	175	210	174	211	77	52
17	144	242	111	243	110	20	141	177	208	82	47	83	46	180	205
236	117	11	150	10	151	233	120	76	53	171	214	170	215	73	56
13	148	238	115	239	114	16	145	173	212	78	51	79	50	176	209
248	105	23	138	22	139	245	108	88	41	183	202	182	203	85	44
25	136	250	103	251	102	28	133	185	200	90	39	91	38	188	197
228	125	3	158	2	159	225	128	68	61	163	222	162	223	65	64
5	156	230	123	231	122	8	153	165	220	70	59	71	58	168	217
256	97	31	130	30	131	253	100	96	33	191	194	190	195	93	36

Magic [3]

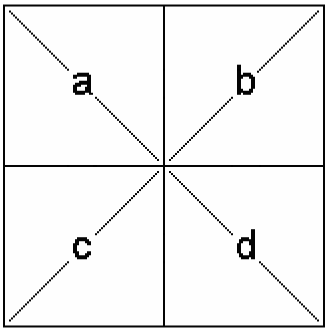
12x12 Franklin Squares

Cor A. J. Hurkens [3] [4] determined experimentally by an exhaustive search in 3.5 hours with a network of 50 computers in parallel; equivalent in one computer to a total computation time of approximately 165 hours that the *12x12 Franklin Squares* does not exist. No *Semi-Magic,* no *Magic!*

A Partial Demonstration

The algebraic demonstration of the nonexistence of magic solutions for 12x12 Franklin Squares and in general for the orders n = 8k + 4; is not complicated:

1) *In the Magic Franklin Squares, the quadrants are also magic; with diagonals that add half of the magic sum*:



For any *Franklin Square*:

$$a + b = a + c \quad ; \quad \text{then } b = c$$
$$b + a = b + d \quad ; \quad \text{then } a = d$$

In a *Magic Franklin Square*:

$$a + d = b + c = M_{s(n)}$$

then:

$$a = b = c = d = M_{s(n)} / 2$$

2) *In each main diagonal of a semi-magic or magic square of order n = 2k (k ≥ 2) with the entries of the 2x2 sub-squares that add a common sum, is verified two sets of alternate entries that add a common sum*:

As examples of this property, it will be demonstrated for the orders 4 and 6:

a) For a square of order n = 4:

$$\begin{matrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{matrix}$$

$$n_{11} + n_{21} + n_{31} + n_{41} = n_{41} + n_{42} + n_{43} + n_{44} \qquad \textbf{(1)}$$

By the property of the 2x2 sub-squares:

$$n_{21} + n_{31} = n_{23} + n_{33} \text{ and } n_{42} + n_{43} = n_{22} + n_{23} \qquad \textbf{(2)}$$

Replacing **(2)** in **(1)** and simplifying:

$$n_{11} + n_{23} + n_{33} + n_{41} = n_{41} + n_{22} + n_{23} + n_{44}$$

Then:

$$n_{11} + n_{33} = n_{22} + n_{44}$$

For the other diagonal:

$$n_{14} + n_{24} + n_{34} + n_{44} = n_{41} + n_{42} + n_{43} + n_{44} \quad \textbf{(1)}$$

By the property of the 2x2 sub-squares:

$$n_{24} + n_{34} = n_{22} + n_{32} \text{ and } n_{42} + n_{43} = n_{22} + n_{23} \quad \textbf{(2)}$$

Replacing **(2)** in **(1)** and simplifying:

$$n_{14} + n_{22} + n_{32} + n_{44} = n_{41} + n_{22} + n_{23} + n_{44}$$

Then:

$$n_{14} + n_{32} = n_{41} + n_{23}$$

b) For a square of order n = 6:

$$\begin{matrix} n_{11} & n_{12} & n_{13} & n_{14} & n_{15} & n_{16} \\ n_{21} & n_{22} & n_{23} & n_{24} & n_{25} & n_{26} \\ n_{31} & n_{32} & n_{33} & n_{34} & n_{35} & n_{36} \\ n_{41} & n_{42} & n_{43} & n_{44} & n_{45} & n_{46} \\ n_{51} & n_{52} & n_{53} & n_{54} & n_{55} & n_{56} \\ n_{61} & n_{62} & n_{63} & n_{64} & n_{65} & n_{66} \end{matrix}$$

$$n_{11} + n_{21} + n_{31} + n_{41} + n_{51} + n_{61} = n_{61} + n_{62} + n_{63} + n_{64} + n_{65} + n_{66} \quad \textbf{(1)}$$

By the property of the 2x2 sub-squares:

$$\begin{matrix} n_{21} + n_{51} = n_{25} + n_{55} \text{ and } n_{31} + n_{41} = n_{33} + n_{43} \\ n_{62} + n_{65} = n_{22} + n_{25} \text{ and } n_{63} + n_{64} = n_{43} + n_{44} \end{matrix} \quad \textbf{(2)}$$

Replacing **(2)** in **(1)** and simplifying:

$$n_{11} + n_{25} + n_{33} + n_{43} + n_{55} + n_{61} = n_{61} + n_{22} + n_{43} + n_{44} + n_{25} + n_{66}$$

Then:

$$n_{11} + n_{33} + n_{55} = n_{22} + n_{44} + n_{66}$$

For the other diagonal:

$$n_{16} + n_{26} + n_{36} + n_{46} + n_{56} + n_{66} = n_{61} + n_{62} + n_{63} + n_{64} + n_{65} + n_{66} \quad \textbf{(1)}$$

By the property of the 2x2 sub-squares:

$$\begin{matrix} n_{26} + n_{56} = n_{22} + n_{52} \text{ and } n_{36} + n_{46} = n_{34} + n_{44} \\ n_{62} + n_{65} = n_{22} + n_{25} \text{ and } n_{63} + n_{64} = n_{43} + n_{44} \end{matrix} \quad \textbf{(2)}$$

Replacing **(2)** in **(1)** and simplifying:

$$n_{16} + n_{22} + n_{34} + n_{44} + n_{52} + n_{66} = n_{61} + n_{22} + n_{43} + n_{44} + n_{25} + n_{66}$$

Then:

$$n_{16} + n_{34} + n_{52} = n_{61} + n_{43} + n_{25}$$

3) *For the hypothetical Magic Franklin Squares of order n = 8k + 4, the diagonals of the quadrants add an odd number:*

k	*n = 8k + 4*	$Ms_{(n)}$	$Ms_{(n)} / 2$
0	4	34	17
1	12	870	435
2	20	4010	2005
3	28	10990	5495

4) *For these orders, dividing the diagonal of the quadrants in two for to obtain the value of the sets that add a common sum is obtained a fractional number, in consequence there is not solution.-*

For any *8x8 Franklin Squares* is possible the following transformation [7)]:

Magic ⇔ Operation ⇔ *Semi-Magic*

For the *12x12 Franklin Squares:* of the nonexistence of *Magic* solutions, can be inferred the nonexistence of *Semi-Magic* solutions? Based in the following result, the answer is negative:

20x20 Franklin Squares

The *20x20 Franklin Squares* don't have magic solution however has been obtained *Semi-Magic:*

1	398	2	397	11	396	12	388	14	386	15	387	13	389	5	390	4	399	3	400
395	8	394	9	385	10	384	18	382	20	381	19	383	17	391	16	392	7	393	6
61	338	62	337	71	336	72	328	74	326	75	327	73	329	65	330	64	339	63	340
335	68	334	69	325	70	324	78	322	80	321	79	323	77	331	76	332	67	333	66
81	318	82	317	91	316	92	308	94	306	95	307	93	309	85	310	84	319	83	320
315	88	314	89	305	90	304	98	302	100	301	99	303	97	311	96	312	87	313	86
221	178	222	177	231	176	232	168	234	166	235	167	233	169	225	170	224	179	223	180
200	203	199	204	190	205	189	213	187	215	186	214	188	212	196	211	197	202	198	201
121	278	122	277	131	276	132	268	134	266	135	267	133	269	125	270	124	279	123	280
275	128	274	129	265	130	264	138	262	140	261	139	263	137	271	136	272	127	273	126
141	258	142	257	151	256	152	248	154	246	155	247	153	249	145	250	144	259	143	260
255	148	254	149	245	150	244	158	242	160	241	159	243	157	251	156	252	147	253	146
181	218	182	217	191	216	192	208	194	206	195	207	193	209	185	210	184	219	183	220
240	163	239	164	230	165	229	173	227	175	226	174	228	172	236	171	237	162	238	161
101	298	102	297	111	296	112	288	114	286	115	287	113	289	105	290	104	299	103	300
295	108	294	109	285	110	284	118	282	120	281	119	283	117	291	116	292	107	293	106
41	358	42	357	51	356	52	348	54	346	55	347	53	349	45	350	44	359	43	360
355	48	354	49	345	50	344	58	342	60	341	59	343	57	351	56	352	47	353	46
21	378	22	377	31	376	32	368	34	366	35	367	33	369	25	370	24	379	23	380
375	28	374	29	365	30	364	38	362	40	361	39	363	37	371	36	372	27	373	26

20x20 Semi-Magic Franklin Square obtained by Huub Reijnders [3) 5)]

Conclusion

The magic squares of order n = 4k + 2 with 2x2 sub-squares = 2 (n^2 + 1) and the Magic Franklin Squares of order n = 8k + 4; does not exist.

Question

The nonexistence of the *4x4 Franklin Squares* is easily demonstrated [4)]; now we have the algebraic demonstration for the nonexistence of the *Magic Franklin Squares* of order n = 8k + 4; then:

¿Is possible an algebraic demonstration for the nonexistence of 12x12 Semi-Magic Franklin Squares...?

References:

1) Schindel D., Rempel M. and Loly P. "Enumerating the bent diagonal squares of Dr Benjamin Franklin FRS". January **2006**.-

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4) Hurkens C.A.J. "Constructing Franklin Magic Squares". October **2007**.-

5) Arno Van den Essen "De vruchten van een hype: nieuwe en onmogelijke Franklin vierkanten". June **2007**.-

6) Amela M.A. "Structured 8x8 Franklin Squares", May **2006**.-

7) Amela M.A. "The Canadian conjecture on the 8x8 Franklin Squares", September **2006**.-

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